3.3, 3.5, part of 3.8: Basic Differentiation Rules

Constants, Sums and Differences

(0) $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(f \pm g)(x) = f'(x) \pm g'(x)$.

$\frac{d}{dx}(cf(x)) = cf'(x)$.

Powers

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof: $\lim_{z \to x} \frac{f(z)-f(x)}{z-x} = \lim_{z \to x} \frac{z^n - x^n}{z-x} = \lim_{z \to x} \frac{(z-x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z-x}$

$= nx^{n-1}$.

or pascal's triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Ex: $\frac{d}{dx}(x^3) = 3x^2$, $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{-1/3}$, $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}\,x^{\sqrt{2}-1}$

$\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-5} = -\frac{4}{x^5}$, $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-7/3}$

$\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{\frac{2+\pi}{2}}\right) = \frac{2+\pi}{2}x^{\frac{2+\pi}{2}-1} = \frac{2+\pi}{2}x^{\frac{\pi}{2}}$.

Ex(2): Does $y=x^4-2x^2+2$ have any horizontal tangent lines?

$\frac{d}{dx}(x^4-2x^2+2) = 4x^3-4x$. Hor. tangent line means this is 0

$= 4x(x^2-1) = 0$ when $x=0,-1,1$. So yes.

Exponential:

(2) $\frac{d}{dx}(e^x) = e^x$

Proof: $\lim_{h\to 0} \frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} \frac{e^h-1}{h} \cdot e^x = e^x$

b/c we defined e to be the value st e^x has slope 1 at $x=0$.

Ex(3): Find the equation of the tangent line to $f(x)=e^x$ which goes through the origin.

We want $y=mx$ to be a tangent line at $x=a$.

$f'(a)=e^a$ and thus $m=a$ and goes through point (a, e^a)

Therefore the tangent line through $x=a$ is

$$y-e^a = e^a(x-a) \quad \text{or} \quad y=e^a x - e^a a + e^a.$$

So $e^a - ae^a = e^a(1-a) = 0$ when $a=1$.

So the tangent line to $f(x)$ at $x=1$ is given by

$$y = ex.$$

(4) Product Rule: $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$ or

Let $u = f(x)$ and $v = g(x)$.

$$\frac{d}{dx}(f \cdot g(x)) = f'(x)g(x) + f(x)g'(x).$$

Ex(4): $\cancel{\frac{d}{dx}}$ $y = \frac{1}{x}(x^2+e^x) = \cancel{f(x)}\, u \cdot v$ where $u = \frac{1}{x}$ and $v = x^2+e^x$.

Thus $\frac{d}{dx}\left(\frac{1}{x}(x^2+e^x)\right) = \frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx} = -\frac{1}{x^2}(x^2+e^x) + \frac{1}{x}(2x+e^x)$

$$= 1 + e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$$

$$= 1 + e^x\left(\frac{x-1}{x^2}\right).$$

Ex(5): $y = e^{2x} = e^x \cdot e^x = f(x) \cdot g(x)$ where $f(x) = e^x$ and $g(x) = e^x$.

Thus $\frac{dy}{dx} = \frac{d}{dx}(f \cdot g(x)) = f'(x)g(x) + f(x)g'(x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$.

Ex(6): $\frac{d}{dx}\left((x^2+1)(x^3+3)\right) = 2x(x^3+3) + (x^2+1) \cdot 3x^2 = 5x^4 + 3x^2 + 6x.$

Proof of Product Rule:

$$\frac{d}{dx}(f \cdot g(x)) = \cancel{f(x+h)} \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)[g(x+h) - g(x)]}{h} + \lim_{h \to 0} \frac{g(x)(f(x+h) - f(x))}{h} = f(x)g'(x) + f'(x) \cdot g(x).$$

(6) Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{low \cdot dhi - hi \cdot dlow}{low^2}$

$u = f(x),\ g(x) = v$

or $\frac{d}{dx}\left(\frac{f}{g}(x)\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$.

Ex(7): (a) $\frac{d}{dx}\left(\frac{t^2-1}{t^3+1}\right) = \frac{d}{dx}\left(\frac{u}{v}\right)$ where $u = t^2-1$ and $v = t^3+1$

$$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(t^3+1)(2t) - (t^2-1)(3t^2)}{(t^3+1)^2}$$

(b) $\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{d}{dx}\left(\frac{f}{g}(x)\right)$ where $f(x) = 1$ and $g(x) = e^x$.

Thus $= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} = \frac{e^x \cdot 0 - 1 \cdot e^x}{e^{2x}} = -\frac{e^x}{e^{2x}} = -\frac{1}{e^x}$.

Ex(8): $\frac{d}{dx}\left(\frac{(x-1)(x^2-2x)}{x^4}\right) = \frac{x^4\frac{d}{dx}((x-1)(x^2-2x)) - (x-1)(x^2-2x)\frac{d}{dx}(x^4)}{(x^4)^2}$

$$= \frac{x^4(1\cdot(x^2-2x) + (x-1)(2x-1)) - (x-1)(x^2-2x)\cdot 4x^3}{x^8}$$

Proof of Quotient Rule:

$\frac{d}{dx}\left(\frac{f}{g}(x)\right) = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{g(x)f(x+h) - f(x)g(x+h)}{h\cdot g(x+h)g(x)} = \lim_{h\to 0}\frac{g(x)f(x+h) - g(x)f(x) + g(x)f(x) - f(x)g(x+h)}{h\cdot g(x+h)g(x)}$

$$= \lim_{h\to 0}\frac{g(x)[f(x+h)-f(x)]}{h\cdot g(x+h)g(x)} + \lim_{h\to 0}\frac{f(x)[g(x)-g(x+h)]}{h\, g(x+h)g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$